2019 John O'Bryan Mathematical Competition
Freshman-Sophomore Individual Test

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. **Exact** answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

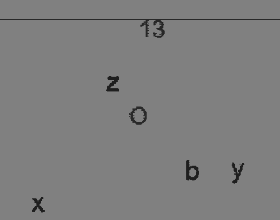
1. For all real values of x, $6x^2+bx+c=3\left(ax^2+3x+4\right)$. Determine the sum $\left(a+b+c\right)$.

2. The sum of the mode and median is k larger than the arithmetic mean for the set of numbers $\{2, 2, 8, 2, 0, 1, 5\}$. Determine the value of k. Express your answer as a common or improper fraction reduced to lowest terms.

3. A triangle with sides extended is shown (but not necessarily drawn to scale). It

[redacted]

has angle measures as labeled. $a:b:c=5:7:8$. Determine the ratio of $x:y:z$. Express your answer in the form $x:y:z$.

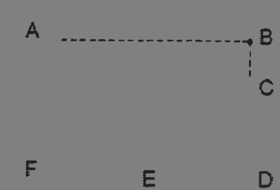


4. A prime number less than 20 is randomly selected. Determine the probability that the number selected is even. Express your answer as a common fraction

x and 30.

9. x is a positive odd integer. Determine *the sum* of all the distinct value(s) of x such that $-(x-11)=|x-11|$

10. A triangular piece broke off rectangle $ABDF$ leaving trapezoid $ACDF$. BD 16, $BC=7$, $FD=24$, and E is the midpoint of $\overline{FD}$. Find the perimeter of ΔACE.



Name: Team Code:

2019 John O'Bryan Mathematical Competition
Freshman/Sophomore Individual Test

Note: All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. [illegible]

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20.

2019 John O'Bryan Mathematical Competition

[redacted]

Must be this exact decimal.

[redacted]

3. 15:13:12 — Must be exactly this format.

13.

Must be this reduced fraction.

171 (feet optional)

5.

15. $\frac{40}{49}$ — Must be this reduced fraction.

6. 13,680 (people optional)

16. 75

7. $\frac{169}{12}$ — Must be this improper fraction.

17. $\frac{1}{2}$ — Must be this reduced fraction.

8. 19

18. 17

9. 36

19. 2.580 — Must be this exact decimal (trailing zero is necessary).

10. 60

20. (6,8) — Must be this ordered pair.

Awards Lists and Solutions to the Team Competition may be found at http://math.nku.edu/job

2019 John O'Bryan Mathematical Competition
Junior-Senior Individual Test

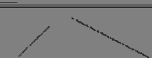
Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of [redacted]

3. Determine the number of distinct prime factors for the number 426,888.

4. Determine the number of rectangles in the figure on the right.

5. Jordan drove to a town 25 miles away at an average speed of 50 miles per hour. The return trip along the same route took 20 minutes longer than the trip to town. Determine the average speed in miles per hour for Jordan's round trip. Express your answer as an exact decimal.

[redacted]



1

7. Determine the exact distance from the center of the circle given by $x^2 + 8x + y^2 - 6y + 3 = 2$ to the [redacted]

10. Suppose $x = \quad 20 + \quad 20 + \quad 20 + \cdots$ Determine the exact value of x.

Name: Team Code:

2019 John O'Bryan Mathematical Competition
Junior/Senior Individual Test

Note: All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1.
11.
2.
12.
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Name: _______ANSWERS_____ Team Code:

2019 John O'Bryan Mathematical Competition

[illegible]

1.	4	
2.	15	
3.	4	
4.	100	
5.	37.5	(miles per hour optional units)
6.	$\frac{5}{6}$	Must be this reduced fraction.
7.	$2\sqrt{5}$	Must be this simplified form.
8.	885.8	Must be exactly this answer.
9.	$\frac{10}{3}$	Must be this simplified form.
10.	5	
11.	$\frac{1}{2}$	Must be this reduced fraction.
12.	$(3,1)$	Must be this ordered pair.
13.	$10\sqrt{59}$	Must be this simplified form.
14.	$\frac{6}{5}$	Must be this reduced fraction.
15.	3	
16.	58.73	Must be exactly this answer.
17.	2.58	Must be exactly this answer.
18.	144	
19.	48	
20.	7	

2019 John O'Bryan Mathematical Competition
Questions for the Two-Person Speed Event

Calculators ma not be used on the first four uestions

1. Let $k = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac}$ with $a + b + c = 15$ and $abc = 5$. Let $w = \frac{z}{xy} + \frac{y}{xz} + \frac{x}{yz}$ with $x^2 + y^2 + z^2 = 12$ and $xyz = 3$. Determine $(k + w)$.

2. Let $f(x) = sin^2(19x) + cos^2(19x)$ and $g(x) = \begin{vmatrix} 2 & x \\ -1 & 9 \end{vmatrix}$. If $k = f(g(2))$ and $w = g(f(2))$, then determine $(k + w)$.

3. Let $\log_8 64 + \log_2 k - 2\log_4 8 = 4$ and $3\sin\frac{\pi}{6} + 4\tan\theta\pi + 7\cos\frac{5\pi}{3} = 1$ with $0 \leq \theta \leq 1$. Determine $\frac{k}{\theta}$.

4. Let s be the coefficient of x^4 in $(x+1)^7$ and t be the coefficient of y^3 in $(y+2)^5$ Determine $(s - t)$.

Calculators ma be used on the remainin uestions

5. Let a be a positive integer number base so that $28_a = 132_5$ Let $3^{b-2c} = \frac{3^{2b-c}}{81}$. Determine $(a + b + c)$.

[redacted]

Names: ______________________________ School: ______________________

2019 John O'Bryan Mathematical Competition

Answers for the Two-Person Speed Event

[redacted]

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless [redacted]

1.

[redacted]

2.

[redacted]

3.

[redacted]

4.

[redacted]

5.

6.

7.

8.

SCORE

T1.

T2.

3. **24**

−5

2019 John O'Bryan Mathematics Competition :: 5-Person Team Test

[redacted]

Questions will not be scored without the following two things:

- Place your team code in the upper right corner of each page that will be turned in.
- Place question numbers in the upper left corner of each page that will be turned in.

Questions are equally weighted. Teams must **show complete solutions (not just answers)** to receive credit. More specific instructions are read verbally at the beginning of the test.

[redacted]

C

Figure 1: Scenario described in Question 4

2019 John O'Bryan Mathematics Competition :: 5-Person Team Test

1. Let $f(\ )\quad {}_0 + {}_1\ + {}_2\ {}^2 + \cdots + {}_6\ {}^6$ be an arbitrary polynomial of degree 6. Define

$$(\ )\quad f(\ ) - f.$$

   (a) Find $(\ )$ and (f). Note: your answer may include the constant coefficients ${}_0,\ {}_1$

2. A non-negative integer is a *square number* if ^{2} for some integer .

(a) Determine how many positive integers with $\leq$ can be written as the sum of two square numbers (not necessarily distinct). That is, $^2 +\ ^2$ for some integers and .

> **Solution:** Note that 0, 1, 4, 9, and 16 are the squares that are at most 20. There are $\binom{5}{2}$ $\frac{5\cdot 4}{2}$ possible sums of distinct squares (only one of which is more than 20), and 5 possible sums of the same square (two of which are not in the interval ,). This gives a maximum of $+$ sums of two squares. Listing the possible positive sums to check for repeats gives
>
> , , , , , , , , , , , .
>
> Therefore, there are 12 such positive integers.

(b) Assume $^2 +\ ^2 +\ ^2$ with $>$. Note that $^2 +\ ^2 -\ ^2 >$. Prove that ^{2} is the sum of three, positive square numbers.

> **Solution:** Let $^2 +\ ^2 +\ ^2$. Expanding and rearranging gives
>
> $$(\ ^2 +\ ^2 +\ ^2)^2 \quad ^4 +\ ^4 +\ ^4 +\ ^2\ ^2 +\ ^2\ ^2 +\ ^2\ ^2$$
> $$\left(\ ^4 +\ ^4 +\ ^4 +\ ^2\ ^2 -\ ^2\ ^2 -\ ^2\ ^2\right) +\ ^2\ ^2 +\ ^2\ ^2$$
> $$\left(\ ^2 +\ ^2 -\ ^2\right)^2 + (\quad)^2 + (\quad)^2,$$
>
> which is the sum of three squares.

3. Let ( ) be the function that sums the digits of a positive integer ; e.g. ( ) + + .

(a) Determine + ( ) + (( )) when .

> **Solution:** Note ( ) + + + . Then + ( ) + (( )) + + .

(b) Consider M where M + ( ) + (( )) for some positive integer . Prove that M is a multiple of an integer K with $K >$.

> **Solution:** Note that a positive integer is divisible by 3 exactly when the sum of its digits is divisible by 3. Thus $\equiv$ ( ) $\equiv$ (( )) . Hence, + ( ) + (( )) $\equiv$ $\equiv$. Therefore, M must be a multiple of 3.

4. Let A, B, C be vertices of a triangle with angles , , (see Figure 1). Let be the center of the circle outside of ABC tangent to $\overline{BC}$, $\overrightarrow{AB}$, and $\overrightarrow{AC}$. Note that is the intersection of the bisector of the angle A, and the bisector of the exterior angle at vertex B.

   (a) Determine \ A B in terms of .

   Solution: Notice that \ BA $\frac{a}{2}$, \ CB $\frac{180^\circ - b}{2}$ and \ AB + \ CB $^\circ + \frac{b}{2}$. From the angle sum for A B and ABC we have

   $$\backslash\ A \quad B \quad 1$$

5. At the start of the school year Jamie has 18 classmates in maths class, for a total of 19 students in the class. Each of Jamie's 18 classmates has a unique number of friends in the class (which could include Jamie). For example, Alex and Riley can **not**

6. Consider the operation of *square-replacement*, in which a square is replaced with four equal-sized squares (i.e. splitting the square into quarters).

(a) Starting from the unit square, consider the iterative process of square-replacing every square. Let $\alpha(\ )$ denote the sum of perimeters for *all* squares after iterations. Determine $\alpha(\ )$ for ,, , and general .

Figure 2: Iterations of square replacement, having 1, 4, 16, and 64 squares, respectively.

> **Solution:** After iterations, there are $(\ )^2$ squares each with side length $^-$. Thus
>
> $$\alpha(\ ) \quad ^2 \ (\ \cdot \ ^-) \qquad ^{+2}.$$
>
> So $\alpha(\ )$, $\alpha(\ )$, $\alpha(\ )$, and $\alpha(\ )$.

(b) Continuing from part (a), let $\delta(\ )$ denote the sum of perimeters of squares that overlap the main diagonal of the unit square (possibly at a corner) after iterations. Determine $\delta(\ )$ for ,, , and general .

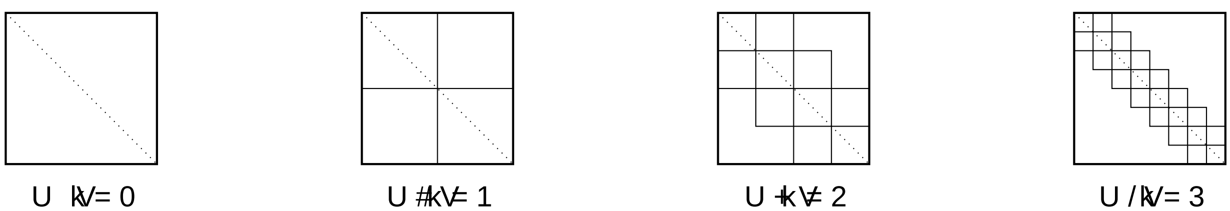

Figure 3: Squares contributing to $\delta(\ )$, having 1, 4, 10, and 22 squares, respectively.

> **Solution:** After -iterations, there are squares whose diagonal lies on the main diagonal of the unit square and $\cdot (\ \ -\ )$ squares that overlap the main diagonal only at a corner. The side lengths of these squares is still $^-$. Thus
>
> $$\delta(\ ) \quad (\ \cdot \ ^-) (\ \ + \ (\ \ - \ )) \quad - \ ^{3-}.$$
>
> So $\delta(\ )$, $\delta(\ )$, $\delta(\ )$, and $\delta(\ )$.

(c) Is it possible to square-replace in a way that $\delta(\ ) >$ for some ?

> **Solution:** Yes. Note that $\delta(\ ) <$ for square-replacing every square, which does not work. Square-replacing the unit square gives four squares. We call the two squares with sides $\frac{1}{2}$ and overlapping the main diagonal only at a corner level-1 squares, and then square-replace the remaining squares. Next, we call the four squares with sides $\frac{1}{4}$ and overlapping the main diagonal only at a corner level-2 squares, and then square-replace the remaining squares

(leaving the level-1 squares in tact). We continue in this way until we have 505 levels of squares. There are squares at level , each with side $^{-}$. Thus, the total perimeter of all level- squares is 4, and the total perimeter of all squares overlapping the diagonal is $\cdot$ $>$, as desired.